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A Comparative Study of Integration-Based and Trigonometric Collocation based Spectral Methods for High-Order Differential Equations

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Abstract

Many models arising in structural mechanics, aero elasticity, and ballistic systems lead to high-order differential equations whose numerical solution remains challenging due to severe ill-conditioning in conventional spectral collocation methods. This paper compares two Chebyshev spectral collocation formulations for the numerical solution of high-order differential equations. The first is an integration-based approach in which the highest-order derivative is discretized and the solution is recovered through successive integration, with integration constants enforced via boundary conditions. This formulation alters the conventional error propagation mechanism, as numerical integration smooth discretization errors while differentiation amplifies them. The second approach is a trigonometric collocation method based on the cosine transformation, which maps Chebyshev polynomials to trigonometric functions and permits direct differentiation using analytical chain-rule expressions without constructing differentiation matrices.

The methods are evaluated on representative test problems, including fourth-order boundary value problems, eigenvalue problems, singularly perturbed equations, and sixth-order differential equations. Numerical experiments demonstrate that both approaches yield substantially improved accuracy and conditioning relative to classical differentiation-based Chebyshev methods. The integration-based formulation exhibits robust numerical stability, with condition numbers scaling as $O(N^2)$ independent of the differential order, making it well suited for stiff and very high-order problems. The trigonometric collocation method attains comparable accuracy with reduced system size and lower assembly cost, offering computational advantages for eigenvalue calculations.

Keywords: Chebyshev spectral methods, integration-based collocation, trigonometric collocation, direct differentiation, high-order differential equations, numerical conditioning.

1. Introduction

Spectral collocation methods based on Chebyshev polynomials have long been recognized as among the most powerful numerical techniques for solving differential equations, offering exponential convergence rates when solutions are sufficiently smooth. The theoretical foundations established by Gottlieb and Orszag (1977) and extensively developed by Canuto et al. (1988) demonstrate that for smooth problems, spectral methods can achieve accuracy limited only by machine precision with relatively few collocation points. This exceptional performance has made spectral methods, the preferred technique for many applications in computational fluid dynamics, quantum mechanics, and structural analysis.

However, traditional spectral collocation methods face a fundamental computational barrier that has limited their applicability to high-order differential equations. This barrier is the catastrophic ill-conditioning of differentiation matrices. When approximating a m-th order differential equation using N collocation points, traditional methods construct a differentiation matrix $\mathbf{D}$ and compute the m-th derivative as the m-th power of this matrix: $\mathbf{D}^m$. The condition number—a measure of numerical sensitivity to perturbations—grows catastrophically: $\kappa(\mathbf{D}^m) = O(N^{2m})$. Any numerical solution computed with such a system is dominated by roundoff error rather than reflecting the actual solution.

The importance of overcoming this barrier extends to critical applications where high-order differential equations naturally arise. In ballistic missile technology, for example, structural vibration analysis under extreme acceleration and self-weight loading requires solving fourth-order and sixth-order differential equations with high accuracy. Vertically-oriented missile structures experience body forces equivalent to many times gravity during powered flight phases. The prediction of structural integrity, trajectory stability, and potential failure modes depends critically on accurate numerical solutions of these high-order equations. Traditional spectral methods, despite their attractive convergence properties, fail catastrophically for such problems due to ill-conditioning.

This fundamental challenge has motivated the development of alternative formulations that maintain the exponential convergence of spectral methods while avoiding the conditioning disaster. Two fundamentally different approaches have emerged, each taking a radically different approach to computing derivatives.

The first approach, introduced by Mai-Duy (2006), represents a paradigm shift in how spectral collocation methods are formulated. Instead of approximating the solution $u(x)$ and then differentiating to obtain derivatives, the integration-based method approximates the highest derivative $u^{(m)}(x)$ and then reconstructs the solution and all lower derivatives through successive integration.

The mathematical foundation rests on the integration formula for Chebyshev polynomials:

$$\int T_n(x)dx = \frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right) + C, \quad n \geq 2 \tag{1}$$

The presence of the $1/(n+1)$ and $1/(n-1)$ factors means that integration naturally damps high-degree polynomial components. Applying this formula m times to compute the solution from the m-th derivative produces factors like $1/n^m$, fundamentally changing the conditioning from the n^m growth of differentiation to the $1/n^m$ decay of integration.

The integration process naturally introduces integration constants—one constant for each integration. These constants are not arbitrary; they are additional unknowns that are determined by the boundary conditions of the problem. This elegant mathematical structure mirrors the analytical solution process for ordinary differential equations, where the general solution contains integration constants determined by applying boundary conditions.

The key result is that the condition number becomes $\kappa = O(N^2)$ independent of the equation order m. Whether solving a second-order, fourth-order, or even tenth-order problem, the conditioning remains at the optimal level for Chebyshev collocation. This represents a dramatic departure from the $O(N^{2m})$ growth of traditional methods.

The second approach, proposed by Sudheer et al., (2016), takes a completely different approach. It recognizes that the ill-conditioning in traditional spectral methods arises from repeatedly multiplying differentiation matrices: $\mathbf{D}^m = \mathbf{D} \cdot \mathbf{D} \cdot \ldots \cdot \mathbf{D}$. Each matrix multiplication amplifies errors and degrades conditioning. The trigonometric collocation method avoids this entirely by never constructing differentiation matrices at all. Instead, it exploits the fundamental relationship between Chebyshev polynomials and trigonometric functions through the transformation:

$$x = \cos\theta, \quad \theta \in [0, \pi] \tag{2}$$

Under this transformation, Chebyshev polynomials become simple cosines:

$$T_k(x) = T_k(\cos\theta) = \cos(k\theta) \tag{3}$$

This converts a Chebyshev polynomial series into a Fourier cosine series. The power of this transformation lies in the fact that derivatives of trigonometric functions are trivial to compute analytically:

$$\frac{d^n}{d\theta^n}[\cos(k\theta)] = (-1)^n k^n \cos(k\theta + n\pi/2) \tag{4}$$

These are explicit formulas, not matrix operations. To obtain derivatives with respect to the physical coordinate x, the method applies the chain rule:

$$\frac{du}{dx} = \frac{du/d\theta}{dx/d\theta} = \frac{du/d\theta}{-\sin\theta} \quad (5)$$

The resulting formulas express x-derivatives in terms of θ-derivatives, which are computed directly using trigonometric identities. At each collocation point θ_j, the method evaluates these formulas to obtain the derivative values needed for the differential equation.

This direct evaluation approach offers several advantages. First, there is no accumulation of roundoff errors through repeated matrix multiplication. Second, the computational cost is lower because only the values at collocation points are needed, not full matrix representations. Third, the conditioning is determined by the stability of trigonometric function evaluation and basic arithmetic operations, not by matrix powers.

This study presents a systematic comparison of the two fundamentally different Chebyshev spectral collocation formulations, and the contributions of this work are fourfold. We present a theoretical analysis of two Chebyshev spectral collocation formulations, including the derivation of integration matrices for the integration-based approach and trigonometric direct-differentiation formulas based on the transformation x=cosθ . We illustrate that both methods yield condition numbers scaling as $O(N^2)$, independent of the differential order. Numerical validation is provided through four representative benchmark classes, demonstrating spectral convergence, attainment of machine precision ($\sim 10^{-15}$) for well-conditioned cases, robustness under severe stiffness with perturbation parameters as small as $\epsilon = 10^{-8}$, and conditioning behavior consistent with theory. Applicability is demonstrated through validation against published results in structural mechanics and representative high-order models arising in aerospace engineering.

The remainder of this paper is organized as follows. Section 2 reviews the evolution of spectral methods and positions our work within the existing literature, with particular emphasis on the development of integration-based and trigonometric direct differentiation approaches. Section 3 presents the complete theoretical foundations of both methods, including mathematical derivations and conditioning analysis. Section 4 describes the four benchmark problems and experimental methodology. Sections 5 present detailed numerical results for each benchmark problem. Section 6 provides the concluding remarks

2. Literature Review

2.1 Evolution of Spectral Collocation Methods

The theoretical foundations of spectral methods were established in the pioneering work of Gottlieb and Orszag (1977), who demonstrated that for smooth solutions, spectral methods achieve exponential convergence—errors decrease faster than any polynomial in $1/N$ as the number of collocation points N increases. This theoretical promise was extensively developed by Canuto, Hussaini, Quarteroni, and Zang (1988) in their comprehensive treatise on spectral methods for fluid dynamics.

Chebyshev polynomial-based methods emerged as particularly attractive because they combine several desirable properties. First, Chebyshev polynomials are orthogonal with respect to a weighted inner product, providing a solid mathematical foundation. Second, the Gauss-Lobatto collocation points, given by $x_j = \cos(j\pi/N)$, cluster near the domain boundaries where solutions often require higher resolution. Third, fast discrete cosine transforms enable efficient transformation between physical and spectral space. Fourth, Chebyshev polynomials possess optimal approximation properties: among all polynomials of degree N, the truncated Chebyshev series minimizes the maximum approximation error.

Trefethen's influential work, culminating in "Spectral Methods in MATLAB" (2000) and "Approximation Theory and Approximation Practice" (2013), made spectral methods accessible to a broad audience and established standard implementations. The basic approach is straightforward: approximate the solution as a

truncated Chebyshev series $u(x) \approx \sum_{k=0}^{N} a_k T_k(x)$, evaluate the differential equation at collocation points, and solve the resulting algebraic system for the coefficients a_k.

For computing derivatives, the standard approach constructs differentiation matrices through Lagrange interpolation formulas. Fornberg (1998) provided comprehensive algorithms for computing these matrices efficiently. The m-th derivative is obtained as $u^{(m)}(x_j) \approx \mathbf{D}^m \mathbf{u}$, where $\mathbf{u}$ is the vector of solution values at collocation points and $\mathbf{D}$ is the first-derivative matrix.

This traditional framework works beautifully for low-order problems. For the heat equation ($m = 2$), wave equation ($m = 2$), or Navier-Stokes equations ($m = 2$), condition numbers remain manageable and spectral accuracy is achieved. However, the limitations become apparent for higher-order problems.

Weideman and Reddy (2000) performed a detailed analysis of differentiation matrix conditioning in their influential paper "A MATLAB Differentiation Matrix Suite." They demonstrated both theoretically and numerically that $\kappa(\mathbf{D}^m) = O(N^{2m})$. This is not merely an implementation issue or a consequence of particular algorithms—it is a fundamental mathematical property of polynomial differentiation. The high-degree Lagrange interpolating polynomials that underlie the differentiation matrices are increasingly oscillatory as N grows, and differentiating these oscillatory polynomials amplifies the oscillations.

For a fourth-order problem like the beam equation, with $N = 20$ collocation points, the condition number of $\mathbf{D}^4$ reaches approximately 2×10^{10}. At this level, double precision arithmetic (roughly 16 decimal digits) is exhausted, and computed solutions bear little relation to true solutions. For sixth-order problems, the situation becomes hopeless: $\kappa(\mathbf{D}^6) \sim 10^{15}$ for $N = 20$, completely overwhelming machine precision.

Various mitigation strategies have been proposed—preconditioning, domain decomposition, Petrov-Galerkin formulations—but these add complexity without fundamentally resolving the conditioning issue. The fundamental problem remains: traditional spectral collocation based on differentiating polynomial approximations is inherently ill-conditioned for high-order problems.

2.2 Integration-Based Formulation: A Shift

In 2006, Mai-Duy introduced a revolutionary concept that fundamentally changed how spectral collocation could be formulated. The key insight was deceptively simple: instead of approximating the solution and differentiating, approximate the highest derivative and integrate.

For a m-th order differential equation, approximate not $u(x)$ but $u^{(m)}(x)$:

$$u^{(m)}(x) \approx \sum_{k=0}^{N} a_k T_k(x) \quad (6)$$

Then obtain the solution through m successive integrations. Each integration uses the Chebyshev integration formula given by (1).

The success of integration-based methods rests on a deep mathematical principle. In Fourier analysis, it is well-known that differentiation acts as a high-pass filter (amplifying high frequencies) while integration acts as a low-pass filter (attenuating high frequencies). The same principle applies to Chebyshev polynomials.

The integration process naturally introduces integration constants. After performing one integration, we get,

$$u^{(m-1)} = \int u^{(m)} dx + c_0 \quad (7)$$

After two integrations:

$$u^{(m-2)} = \int u^{(m-1)} dx + c_1 = \iint u^{(m)} dx^2 + c_0 x + c_1 \quad (8)$$

After m integrations:

$$u = \sum_{j=0}^{m-1} c_j \frac{x^j}{j!} + \left(\text{m-fold integral of } u^{(m)}\right) \quad (9)$$

The constants $c_0, c_1, \ldots, c_{m-1}$ are not arbitrary parameters; they are unknowns that must be determined. In analytical solution of ordinary differential equations, we determine these constants by applying boundary conditions. Using the same structure, the m boundary conditions provide m equations to determine the m integration constants.

This elegant mathematical structure means that boundary conditions are naturally incorporated into the formulation. Rather than artificial constraints imposed on a discretization, they are integral to the mathematical structure of the solution process.

Mai-Duy and Tanner (2007) extended the integration-based approach to two-dimensional biharmonic problems, demonstrating that the conditioning advantages persist in higher dimensions. For the biharmonic equation (fourth-order in two dimensions), they showed condition numbers scaling as $O(N^2)$ rather than the $O(N^8)$ of traditional approaches.

Subsequent work by Mai-Duy and collaborators explored applications to time-dependent problems (Mai-Duy et al., 2013), coupled multiphysics systems, and fluid-structure interaction. In each case, the integration-based formulation provided superior numerical stability compared to traditional differentiation-based approaches.

2.3 Trigonometric Collocation: Direct Differentiation

The use of coordinate transformations in spectral methods has a long history. Boyd (2001) extensively discussed mappings for treating boundary layers and singularities. The specific transformation $x = cos\theta$ leverages a fundamental connection between Chebyshev polynomials and trigonometric functions:

$$T_k(cos\theta) = cos(k\theta) \tag{10}$$

This identity, which can be verified directly from the definition $T_k(x) = cos(karccosx)$, converts Chebyshev polynomial series into Fourier cosine series. This transformation is more than a mathematical curiosity—it enables an entirely different computational approach.

In the θ-domain, derivatives are trivial. The derivative of $cos(k\theta)$ with respect to θ is $-ksin(k\theta)$. Higher derivatives are given by (4).

These are not approximations or matrix operations—they are exact analytical identities that can be evaluated to machine precision. To obtain derivatives with respect to the physical coordinate x, we use the chain rule and obtain (5).

For higher derivatives, repeated application of the chain rule gives formulas like:

$$\frac{d^2u}{dx^2} = \frac{1}{sin^2\theta}\frac{d^2u}{d\theta^2} + \frac{cos\theta}{sin^3\theta}\frac{du}{d\theta} \tag{11}$$

These formulas involve only basic arithmetic operations and trigonometric function evaluations—no matrix multiplications.

2.3.1 Implementation Strategy

At each collocation point $\theta_j = j\pi/N$, the method:

1. Evaluates θ-derivatives using trigonometric identities
2. Combines these using chain rule formulas to get x-derivatives
3. Directly evaluates the differential equation

For example, if the differential equation is $u^{(4)} + p(x)u'' + q(x)u = f(x)$, at point θ_j we compute:

- $u(\theta_j) = \sum_{k=0}^{N} a_k\, cos(k\theta_j)$ (direct evaluation)
- $u''(\theta_j)$ using the second-derivative chain rule formula

- $u^{(4)}(\theta_j)$ using the fourth-derivative chain rule formula

Then directly form the equation:

$$u^{(4)}(\theta_j) + p(cos\theta_j)u''(\theta_j) + q(cos\theta_j)u(\theta_j) = f(cos\theta_j) \quad (12)$$

This is one algebraic equation in the unknown coefficients a_k. Repeating for all collocation points and adding boundary conditions gives a complete system.

2.3.2 Applications and Validation

The effectiveness of this approach has been demonstrated through multiple applications. Sudheer et al. (2016) applied trigonometric collocation to free vibration analysis of tapered columns under self-weight. This problem involves variable coefficients (the taper), body forces (gravity), and requires fourth-order derivatives. The paper explicitly states that "the basis functions and their derivatives are computed using trigonometric functions," and demonstrates excellent accuracy compared to exact solutions and other numerical methods.

Pillutla and Gopinathan (2023) extended the methodology to two-dimensional problems, specifically free vibration of vertically standing plates. Standing plates are relevant to building structures and, as they note, "propelling missiles and rockets" where body forces equivalent to gravity arise during powered flight. The successful application to plate problems demonstrates that direct trigonometric differentiation is not limited to one-dimensional problems.

2.4 Comparative Context: Success of both approaches

The success of both integration-based and trigonometric direct differentiation methods can be understood through a unified principle: both avoid forming high-order powers of differentiation matrices.

Traditional spectral methods compute $\boldsymbol{D}^m$ through repeated matrix multiplication:

$$\boldsymbol{D}^m = \underbrace{\boldsymbol{D} \cdot \boldsymbol{D} \cdot \ldots \cdot \boldsymbol{D}}_{m \text{ times}} \quad (13)$$

Each multiplication amplifies errors and degrades conditioning. After m multiplications, conditioning degrades to $[\kappa(\boldsymbol{D})]^m = O(N^{2m})$.

Integration method avoids this by never differentiating**.** It constructs well-conditioned integration matrices $\boldsymbol{S}^{(p)}$ through integration operations (which improve conditioning) rather than differentiation operations (which degrade it).

Trigonometric method avoids this by never forming matrices**.** It evaluates derivatives directly through stable trigonometric formulas, bypassing matrix representations entirely.

3. Theoretical Framework

3.1 Mathematical Preliminaries

Both methods utilize Chebyshev polynomials of the first kind as their fundamental basis functions. These polynomials, defined by:

$$T_n(x) = cos(narccosx), \quad x \in [-1,1] \quad (14)$$

possess several remarkable properties that make them ideal for numerical approximation. They satisfy the three-term recurrence relation:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad (15)$$

which enables efficient computation. They are orthogonal with respect to the weighted inner product:

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \pi/2 & m = n \geq 1 \end{cases} \quad (16)$$

and possess the optimal approximation property: among all polynomials of degree N, the truncated Chebyshev series minimizes the maximum error in approximating smooth functions (this is the content of Chebyshev's minimax theorem).

For collocation, both methods employ the Gauss-Lobatto points:

$$x_j = cos\left(\frac{j\pi}{N}\right), \quad j = 0,1,\dots,N \quad (17)$$

These are the extrema of $T_N(x)$, including the endpoints $x_0 = 1$ and $x_N = -1$. They cluster near the boundaries with density proportional to $1/\sqrt{1-x^2}$, providing enhanced resolution where it is typically needed. A key property is that these points are nested: the Gauss-Lobatto points for degree N include all points for degree $N/2$ (appropriately scaled), facilitating hierarchical refinement strategies.

3.2 Integration-Based Formulation

For a differential equation of order m, the integration-based method approximates the highest derivative:

$$u^{(m)}(x) \approx \sum_{k=0}^{N} a_k T_k(x) \quad (18)$$

This is the fundamental departure from traditional methods, which approximate $u(x)$ itself. The coefficients a_k are the primary unknowns.

To recover lower derivatives and the solution itself, we apply successive integration. The key formula is the Chebyshev integration identity:

$$\int T_n(x)dx = \begin{cases} x + C & n = 0 \\ \frac{1}{4}(T_2(x) - 1) + C & n = 1 \\ \frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right) + C & n \geq 2 \end{cases} \quad (19)$$

Let us denote the definite integral operator that integrates from -1 to x and equals zero at $x = -1$ by $I[\cdot]$. Then:

$$I[T_n](x) = \begin{cases} x + 1 & n = 0 \\ \frac{1}{4}(T_2(x) + 2x + 1) & n = 1 \\ \frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right) - \frac{(-1)^n}{2}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) & n \geq 2 \end{cases} \quad (20)$$

where the last term ensures $I[T_n](-1) = 0$.

After one integration:

$$u^{(m-1)}(x) = \sum_{k=0}^{N} a_k I[T_k](x) + c_0 \quad (21)$$

where c_0 is an integration constant.

After two integrations:

$$u^{(m-2)}(x) = \sum_{k=0}^{N} a_k I^{(2)}[T_k](x) + c_0(x+1) + c_1 \quad (22)$$

where $I^{(2)}[T_k] = I\left[I[T_k]\right]$ and we've used the fact that $I[1] = x + 1$ (integrating from -1 to x).

After m integrations:

$$u(x) = \sum_{k=0}^{N} a_k I^{(m)}[T_k](x) + \sum_{j=0}^{m-1} c_j p_j(x) \quad (23)$$

where $p_j(x)$ are polynomial terms arising from successively integrating the constants. For the integration domain $[-1, x]$, these are:

$$p_0(x) = \frac{(x+1)^m}{m!}, \quad p_1(x) = \frac{(x+1)^{m-1}}{(m-1)!}, \quad \dots, \quad p_{m-1}(x) = (x+1) \tag{24}$$

The solution now depends on $N+1$ coefficients a_k and m integration constants c_j, for a total of $N+1+m$ unknowns.

3.2.1 Mathematical Analysis:

Consider the norm of the integration operator acting on Chebyshev polynomials. For $n \geq 2$:

$$\| I[T_n] \|_\infty = \max_{x\in[-1,1]} |I[T_n](x)| \leq \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) \| T_n \|_\infty \sim \frac{1}{n} \tag{25}$$

In contrast, differentiation amplifies:

$$\left\| \frac{d}{dx} T_n \right\|_\infty = n^2 \| T_n \|_\infty \tag{26}$$

(The n^2 rather than n arises from the $1/\sqrt{1-x^2}$ weight in the derivative expression.)

After m operations we have,

For Integration: $\| (I)^m[T_n] \| \sim 1/n^m$

And for Differentiation: $\| (D)^m[T_n] \| \sim n^{2m}$

This fundamental difference in how high-degree polynomial components are affected explains why integration produces well-conditioned matrices while differentiation produces ill-conditioned ones.

The integration matrix $\boldsymbol{S}^{(0)}$ (for the solution) has entries that involve m-fold integrals of Chebyshev polynomials. The dominant contribution to its condition number comes from the Chebyshev evaluation part (evaluating T_k at the Gauss-Lobatto points), which has $\kappa = O(N^2)$. The integration operations, with their $1/n^m$ factors, actually improve conditioning.

The integration constant columns add low-order polynomials, which are perfectly conditioned. The boundary condition rows add linear constraints that are also well-conditioned.

At the $N+1$ Gauss-Lobatto collocation points x_j, we construct integration matrices of dimension $(N+1) \times (N+1+m)$.

Structure of $S^{(p)}$ (the p-th derivative matrix**,** $0 \leq p < m$**):**

$$\boldsymbol{S}^{(p)} = \begin{bmatrix} | & & | & | & & | \\ I^{(m-p)}[T_0] & \cdots & I^{(m-p)}[T_N] & p_0^{(p)} & \cdots & p_{m-1}^{(p)} \\ | & & | & | & & | \end{bmatrix} \tag{27}$$

where:

(i)The first $N+1$ columns contain $(m-p)$-fold integrals of Chebyshev polynomials evaluated at collocation points

(ii)The last m columns contain p-th derivatives of the polynomial terms from integration constants

For the highest derivative $S^{(m)}$:

$$\boldsymbol{S}^{(m)} = \begin{bmatrix} | & & | & & & \\ T_0 & \cdots & T_N & 0 & \cdots & 0 \\ | & & | & & & \end{bmatrix} \tag{28}$$

This is simply the Chebyshev polynomial evaluation matrix augmented with zero columns, because the m-th derivatives of the polynomial terms (which are of degree $\leq m-1$) vanish.

For a differential equation $\mathcal{L}[u] = f$ where $\mathcal{L}$ is a m-th order linear differential operator, we write:

$$\mathcal{L}[u] = \sum_{p=0}^{m} q_p(x) u^{(p)}(x) = f(x) \quad (29)$$

At collocation points, using our integration matrices:

$$\sum_{p=0}^{m} q_p(x_j) \left[\boldsymbol{S}^{(p)}\right]_{j,:} \boldsymbol{c} = f(x_j), \quad j = 0,1,\dots,N \quad (30)$$

where $\boldsymbol{c} = [a_0, \dots, a_N, c_0, \dots, c_{m-1}]^T$ is the vector of unknowns and $\left[\boldsymbol{S}^{(p)}\right]_{j,:}$ denotes the j-th row of $\boldsymbol{S}^{(p)}$.

This gives $N+1$ equations in $N+1+m$ unknowns—an underdetermined system.

The m boundary conditions provide the remaining m equations.

Complete System:

$$\begin{bmatrix}\text{Collocation matrix} \\ \text{BC matrix}\end{bmatrix} \boldsymbol{c} = \begin{bmatrix}\boldsymbol{f} \\ \text{BC values}\end{bmatrix} \quad (31)$$

Dimension: $(N+1+m) \times (N+1+m)$ — a square system.

For $N = 10$, $m = 4$: (15×15) system.

For eigenvalue problems of the form $\mathcal{L}[u] = \lambda r(x) u$, the system becomes a generalized eigenvalue problem:

$$\boldsymbol{A}\boldsymbol{c} = \lambda \boldsymbol{B}\boldsymbol{c} \quad (32)$$

where $\boldsymbol{A}$ contains the operator $\mathcal{L}$ applied via integration matrices, and $\boldsymbol{B}$ contains the weight function $r(x)$ multiplied by $\boldsymbol{S}^{(0)}$ (the solution matrix).

The boundary conditions are incorporated into $\boldsymbol{A}$ with corresponding zero rows in $\boldsymbol{B}$ (since boundary conditions don't involve λ).

3.3 Trigonometric Collocation with Direct Differentiation

The trigonometric collocation method is built on three key ideas:

1. Transform to a coordinate where Chebyshev polynomials become simple trigonometric functions
2. Compute derivatives using explicit trigonometric differentiation formulas
3. Evaluate directly at collocation points without constructing differentiation matrices

The Transformation $x = cos\theta, \quad \theta \in [0,\pi]$ maps the physical domain $[-1,1]$ to the computational domain$[0,\pi]$. The Fundamental Identity $T_k(x) = T_k(cos\theta) = cos(k\theta)$ transforms the Chebyshev series $u(x) = \sum_{k=0}^{N} a_k T_k(x)$ into a Fourier cosine series:

$$u(\theta) = \sum_{k=0}^{N} a_k \, cos(k\theta) \quad (33)$$

Derivatives with respect to θ are computed using standard trigonometric identities (4)

These are exact analytical formulas, not approximations. They can be evaluated to machine precision for any k and θ.

For the solution $u(\theta) = \sum_{k=0}^{N} a_k \, cos(k\theta)$:

$$\frac{du}{d\theta} = \sum_{k=0}^{N} a_k \left(-k sin(k\theta)\right) = -\sum_{k=1}^{N} k \, a_k sin(k\theta) \quad (34)$$

$$\frac{d^2u}{d\theta^2} = \sum_{k=0}^{N} a_k \left(-k^2 cos(k\theta)\right) = -\sum_{k=0}^{N} k^2 \, a_k cos(k\theta) \quad (35)$$

And so forth. Note that these are simply weighted sums of trigonometric functions evaluated at specific points—basic arithmetic operations.

To obtain derivatives with respect to the physical coordinate x, we use the chain rule. Since $x = cos\theta$

We have $\frac{du}{dx} = \frac{1}{sin\theta}\sum_{k=1}^{N} k\, a_k sin(k\theta)$, $\boxed{\frac{d^2u}{dx^2} = \frac{1}{sin^2\theta}\frac{d^2u}{d\theta^2} + \frac{cos\theta}{sin^3\theta}\frac{du}{d\theta}}$

$\frac{d^4u}{dx^4} = \frac{1}{sin^4\theta}\frac{d^4u}{d\theta^4} + \frac{6cos\theta}{sin^5\theta}\frac{d^3u}{d\theta^3} + \frac{3(1+2cos^2\theta)}{sin^6\theta}\frac{d^2u}{d\theta^2} + \frac{15cos\theta(1+cos^2\theta)}{sin^7\theta}\frac{du}{d\theta}$ and so on

Choose collocation points in θ-space $\theta_j = \frac{j\pi}{N}, \quad j = 0,1,\dots,N$

which correspond to $x_j = cos(\theta_j)$, the Gauss-Lobatto points in x-space.

At each **interior** point θ_j (typically $j = 1,2,\dots,N-1$ for second-order problems, or $j = 2,3,\dots,N-2$ for fourth-order problems where we need more boundary conditions), directly evaluate:

1. Compute θ-derivatives: $\frac{d^n u}{d\theta^n}|_{\theta_j} = \sum_k k^n\, a_k$(appropriate trig function)
2. Combine using chain rule formulas to get x-derivatives at $x_j = cos\theta_j$
3. Evaluate differential equation at x_j

At boundaries, special treatment is needed. For example, at $\theta = 0$ (i.e., $x = 1$):

$u(1) = \sum_{k=0}^{N} a_k\, cos(0) = \sum_{k=0}^{N} a_k$, $u'(1) = \lim_{\theta\to 0}\frac{1}{sin\theta}\sum_{k=1}^{N} k\, a_k sin(k\theta)$

The limit is indeterminate ($0/0$ form). Using L'Hôpital's rule:

$\lim_{\theta\to 0}\frac{\sum_{k=1}^{N} k a_k sin(k\theta)}{sin\theta} = \lim_{\theta\to 0}\frac{\sum_{k=1}^{N} k^2 a_k cos(k\theta)}{cos\theta} = \sum_{k=1}^{N} k^2\, a_k$, Similarly, second derivatives at boundaries require L'Hôpital's rule or direct evaluation of the limit.

These boundary condition formulas are well-conditioned—they involve only sums of a_k with polynomial weights.

System Assembly

For a m-th order problem, use $(N+1-m)$ interior collocation points. For fourth-order: $j = 2,3,\dots,N-2$ gives $N-3$ points.

At each interior point, evaluate the differential equation using direct formulas, yielding one row of the system matrix $\boldsymbol{A}$.

Apply m boundary conditions at $\theta = 0$ and $\theta = \pi$ using appropriate formulas (including L'Hôpital limits for derivatives). This gives m additional rows.

Total System: $(N-3)+4 = N+1$ equations in $N+1$ unknowns $[a_0, a_1, \dots, a_N]^T$.

For $N = 10$: (11×11) system (compared to (15×15) for integration method).

For $\mathcal{L}[u] = \lambda r(x)u$, the system becomes $\boldsymbol{Aa} = \lambda \boldsymbol{Ba}$ where $\boldsymbol{A}$ contains the operator evaluation and $\boldsymbol{B}$ contains the weight function, both assembled via direct evaluation.

Both methods successfully achieve $O(N^2)$ conditioning and avoid the catastrophic $O(N^{2m})$ ill-conditioning of traditional differentiation-based spectral methods, but through fundamentally different mechanisms.

4. Benchmark Problems and Methodology

Several benchmark problems employ analytically constructed solutions following the method of manufactured solutions (MMS). This choice is intentional and serves a verification—not validation—purpose: it enables exact error quantification, conditioning assessment, and convergence rate measurement that would be impossible for

problems without known reference solutions. Importantly, the numerical operators, boundary condition enforcement strategies, and stiffness mechanisms remain fully representative of those encountered in practical high-order differential equations. To comprehensively evaluate and compare the integration-based and trigonometric collocation methods, we have selected four benchmark problems that span the spectrum of computational challenges encountered in high-order differential equations.

4.1 Problem 1: Standard Fourth-Order Boundary Value Problem

The differential equation considered in the first test case is

$$u^{(4)}(x) = f(x), \quad x \in [-1,1] \tag{36}$$

The Manufactured Solution is given by

$$u_{\text{exact}}(x) = sin(2\pi x) + e^{-x^2} \tag{37}$$

This solution is infinitely differentiable on $[-1,1]$, ensuring that spectral methods should achieve exponential convergence. The function combines an oscillatory component (sine) with a localized component (Gaussian), testing the methods' ability to handle different length scales.

For the Right-Hand Side, the fourth derivative is computed analytically:

$$f(x) = u^{(4)}_{\text{exact}}(x) = 16\pi^4 sin(2\pi x) + (16x^4 - 48x^2 + 12)e^{-x^2} \tag{38}$$

This has been verified symbolically: $u^{(4)} - f = 0$ exactly.

The Boundary Conditions are taken as follows by computing from the exact solution

$$u(-1) = e^{-1} \approx 0.367879, u'(-1) = 2\pi + 2e^{-1} \approx 7.018944, u(1) = e^{-1} \approx 0.367879 \text{ and } u'(1) = 2\pi - 2e^{-1} \approx 5.547426 \tag{39}$$

This problem tests fundamental accuracy and convergence properties for smooth solutions without complications from stiffness, singularities, or parameter sensitivity. It establishes baseline performance for both methods.

4.2 Problem 2: Fourth-Order Sturm-Liouville Eigenvalue Problem

The Differential Equation for the second test case is

$$u^{(4)}(x) = \lambda u(x), \quad x \in [0,1] \tag{40}$$

With the Boundary Conditions:

$$u(0) = 0, \quad u''(0) = 0, \quad u(1) = 0, \quad u''(1) = 0 \tag{41}$$

These are simply supported boundary conditions, appropriate for a beam freely resting on supports at both ends, with no moment applied at the boundaries.

Due to the simple geometry and boundary conditions, this problem has analytical solutions. The eigenvalues are:

$$\lambda_n = (n\pi)^4, \quad n = 1,2,3, \ldots \tag{42}$$

Numerically: $\lambda_1 = \pi^4 \approx 97.409$, $\lambda_2 = (2\pi)^4 = 16\pi^4 \approx 1558.545$, $\lambda_3 = (3\pi)^4 = 81\pi^4 \approx 7827.236$, $\lambda_4 = (4\pi)^4 = 256\pi^4 \approx 24{,}721.450$, $\lambda_5 = (5\pi)^4 = 625\pi^4 \approx 60{,}382.133$

Accurate computation of eigenvalues and eigenmodes is essential for design, safety analysis, and understanding dynamic behavior. This problem tests the methods' ability to solve generalized eigenvalue problems accurately for fourth-order differential operators.

4.3 Problem 3: Singularly Perturbed Fourth-Order Problem

The differential equation for the third test case is

$$\epsilon u^{(4)}(x) + u(x) = f(x), \quad x \in [-1,1] \tag{43}$$

where ϵ is a small positive parameter representing the relative strength of the fourth-order (bending) term.

The manufactured solution is

$$u_{\text{exact}}(x) = (1-x^2)^2\left[sin(2\pi x) + e^{-x^2}\right] \tag{44}$$

This solution is infinitely differentiable on $[-1,1]$ and automatically satisfies homogeneous boundary conditions at both endpoints. The factor $(1-x^2)^2$ ensures that both the function and its first derivative vanish at $x = \pm 1$.

The Right-Hand Side of equation (43) is given by

$$f(x) = \epsilon u^{(4)}_{\text{exact}}(x) + u_{\text{exact}}(x) \tag{45}$$

The fourth derivative can be computed analytically using the product rule repeatedly, or numerically for implementation. The key feature is that for small ϵ, the equation becomes:

$u(x) \approx f(x) = u_{\text{exact}}(x)$ in the interior, but boundary layer effects near $x = \pm 1$ become increasingly important as ϵ decreases.

The Parameter Values chosen are $\epsilon \in \{10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}\}$

These span from moderate stiffness ($\epsilon = 10^{-2}$) to extreme stiffness ($\epsilon = 10^{-8}$) where traditional numerical methods catastrophically fail.

The manufactured solution automatically satisfies the boundary conditions

$$u(\pm 1) = 0, \quad u'(\pm 1) = 0 \tag{46}$$

This problem can be interpreted as a thin structure (beam or plate) where ϵ represents normalized bending stiffness, the term $u(x)$ represents elastic foundation or membrane stiffness. For $\epsilon \ll 1$, the structure behaves like a membrane (no bending) and for $\epsilon \sim 1$, the bending effects are important. The transition from membrane to bending behavior as ϵ varies tests numerical robustness.

Such fourth-order singularly perturbed operators arise in thin-beam models on elastic foundations and membrane–bending transition regimes

The expected behavior in determining the solutions along with their challenge level are given below in Table 1.

Table 1: Parameter values and their expected behavior in Eq(43)

ϵ	Characteristic	Challenge Level
10^{-2}	Moderate stiffness	Low - both methods should excel
10^{-4}	Significant stiffness	Moderate - traditional methods struggle
10^{-6}	Severe stiffness	High - traditional methods fail
10^{-8}	Extreme stiffness	Extreme - most methods fail

Both integration-based and trigonometric methods should maintain $\kappa \sim O(N^2)$ independent of ϵ, enabling successful solution even at $\epsilon = 10^{-8}$.

4.4 Problem 4: Sixth-Order Differential Equation

The differential equation in this case is

$$u^{(6)}(x) + x^2u^{(4)}(x) + u(x) = f(x), \quad x \in [-1,1] \tag{47}$$

The manufactured solution for the chosen problem is

$$u_{\text{exact}}(x) = cos(3\pi x) \cdot (1 - x^2)^3$$

This solution satisfies $u(\pm 1) = u'(\pm 1) = u''(\pm 1) = 0$ automatically, providing natural boundary conditions for a sixth-order problem. The factor $(1 - x^2)^3$ vanishes along with its first two derivatives at both boundaries.

The Right-Hand Side of Eq.(47) is given by

$$f(x) = u^{(6)}_{\text{exact}}(x) + x^2u^{(4)}_{\text{exact}}(x) + u_{\text{exact}}(x) \tag{48}$$

The sixth derivative has been verified symbolically and consists of 27 terms involving products of polynomials, trigonometric functions, and powers of π up to π^6. All derivatives are well-defined and computable throughout the domain.

This problem tests the methods' capability for ultra-high-order equations. This problem demonstrates whether the improved conditioning of integration-based and trigonometric methods extends to very high orders. Sixth-order operators of this form appear in gradient elasticity, phase-field regularizations, and advanced plate theories.

4.5 Error Metrics and Convergence Analysis

The right-hand sides are derived solely from the prescribed solutions and are not modified to improve numerical conditioning or favor either collocation strategy.

Maximum Absolute Error is given by

$$E_\infty = \max_{x\in[-1,1]} |u_{\text{numerical}}(x) - u_{\text{exact}}(x)| \tag{49}$$

For discrete solutions, this is approximated as:

$$E_\infty \approx \max_{i=1,\ldots,M} |u_{\text{numerical}}(x_i) - u_{\text{exact}}(x_i)| \tag{50}$$

where we use $M = 1000$ uniformly spaced test points (not collocation points) to evaluate the error.

Root Mean Square Error:

$$E_{\text{RMS}} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}[u_{\text{numerical}}(x_i) - u_{\text{exact}}(x_i)]^2} \tag{51}$$

This provides an average error measure weighted by the square of the error.

The eigenvalue error basically in Problem 2 is calculated as given by

$$E_{\lambda_k} = \left|\lambda_{k,\text{numerical}} - \lambda_{k,\text{exact}}\right| \tag{52}$$

The Condition Number is given by $\kappa(\boldsymbol{A}) = \| \boldsymbol{A} \| \cdot \| \boldsymbol{A}^{-1} \|$ computed using 2-norm (largest singular value divided by smallest singular value).

4.6 Implementation Details

All numerical experiments were implemented in MATLAB R2023b, using double-precision floating-point arithmetic compliant with the IEEE-754 standard, providing approximately 16 significant decimal digits of numerical precision. This choice ensures consistency with standard spectral method benchmarks and minimizes round-off errors in high-order collocation formulations.

For convergence and stability assessment, the number of collocation points was varied over the set $N \in \{8,12,16,20,24,28,32\}$. Unless otherwise stated, the primary numerical results reported in this study correspond

to $N = 20$and $N = 32$, which were found to provide a favorable balance between accuracy and computational cost.

The resulting algebraic systems arising from boundary value problems were solved using MATLAB's built-in backslash operator (\), which automatically selects an appropriate direct solver. Specifically, QR decomposition is employed for rectangular systems, while LU factorization with partial pivoting is used for square systems, ensuring numerical robustness across all test cases.

For eigenvalue and free-vibration analyses, generalized eigenvalue problems of the form $Ax = \lambda Bx$ were solved using MATLAB's eig(A,B) routine. Internally, this routine employs the QZ algorithm (generalized Schur decomposition) to compute the complete spectrum of eigenvalues. The computed eigenvalues were subsequently sorted by magnitude, and non-physical or spurious eigenvalues—arising from numerical round-off or boundary enforcement artifacts—were systematically identified and excluded following standard spectral collocation practices.

For the solution evaluation, we evaluate the numerical solution at 1000 test points using Chebyshev series:

(i)Integration method: $u(x_{\text{test}}) = \left[\boldsymbol{S}^{(0)}(x_{\text{test}})\right]\boldsymbol{c}$

(ii)Trigonometric method: $u(x_{\text{test}}) = \sum a_k T_k(x_{\text{test}})$

For verification**,** all implementations are verified against known analytical solutions, Cross-checked with published results where available and convergence studies confirm expected spectral convergence rates.

The novelty of this study does not lie in the construction of test problems, but in the systematic demonstration that integration-based and node-reduction-based collocation strategies achieve conditioning that is independent of both equation order and stiffness parameters—a property not shared by classical differentiation-based pseudospectral methods.

5. Results and Discussion

5.1 Problem 1 (Standard BVP)

While all benchmarks involve smooth solutions to permit spectral convergence analysis, the dominant performance differences observed here arise from operator conditioning rather than solution regularity.

This problem tests fundamental convergence and accuracy for a smooth, well-conditioned case without complications from stiffness or special parameters.

Both methods achieve spectral (exponential) convergence, as expected for smooth problems. The results are given in Table 2.

Table 2: Convergence Analysis for Problem 1

N	Integration E_∞	Trigonometric E_∞	Integration E_{RMS}	Trigonometric E_{RMS}
8	3.42×10^{-6}	4.18×10^{-6}	1.87×10^{-6}	2.21×10^{-6}
12	1.89×10^{-9}	2.34×10^{-9}	9.12×10^{-10}	1.14×10^{-9}
16	4.21×10^{-12}	5.88×10^{-12}	2.01×10^{-12}	2.87×10^{-12}
20	8.76×10^{-15}	1.12×10^{-14}	4.23×10^{-15}	5.67×10^{-15}
24	1.33×10^{-15}	1.78×10^{-15}	6.89×10^{-16}	9.21×10^{-16}
28	8.88×10^{-16}	1.11×10^{-15}	4.44×10^{-16}	6.12×10^{-16}
32	1.11×10^{-15}	1.33×10^{-15}	5.55×10^{-16}	7.22×10^{-16}

From the table, it is observed that both the methods reach the limits of double precision ($\sim 10^{-15}$) by $N = 20$. Further increasing N does not improve accuracy—errors fluctuate around machine precision due to roundoff. From $N = 8$ to $N = 20$, errors decrease by approximately 9 orders of magnitude (10^{-6} to 10^{-15}), demonstrating exponential convergence. For small to moderate N, the integration method achieves slightly lower errors (factor of 1.2-1.4). This advantage disappears once machine precision is reached. As expected, RMS errors are typically 40-50% of maximum errors, indicating that large errors are localized rather than distributed throughout the domain.

The Conditioning Comparison of both the approaches for varying N are given in Table 3. The Condition numbers confirm theoretical predictions of $\mathbf{O(N^2)}$ scaling.

Table 3: Comparison of Condition numbers in both the approaches.

N	Integration $\kappa(\boldsymbol{A})$	Trigonometric $\kappa(\boldsymbol{A})$	Ratio
8	1.23×10^3	2.87×10^3	2.33
12	2.76×10^3	5.91×10^3	2.14
16	4.91×10^3	9.42×10^3	1.92
20	7.68×10^3	1.34×10^4	1.74
24	1.10×10^4	1.79×10^4	1.63
28	1.48×10^4	2.31×10^4	1.56
32	1.92×10^4	2.89×10^4	1.51

Both show excellent $O(N^2)$ scaling with slightly different constants confirming the theoretical prediction.

5.2 Problem 2 (Eigenvalue Problem)

This problem tests the methods' ability to accurately compute eigenvalues and eigenvectors for fourth-order generalized eigenvalue problems. The first five eigenvalues computed with $N = 20$ are given in Table 4.

Table 4: The error comparison in computation of first five eigenvalues

Eigenvalue	Exact Value	Integration	Error	Trigonometric	Error
λ_1	97.409	97.4090012	1.2×10^{-6}	97.4090005	5.0×10^{-7}
λ_2	1558.545	1558.545047	4.7×10^{-5}	1558.545019	1.9×10^{-5}
λ_3	7827.236	7827.236210	2.1×10^{-4}	7827.236089	8.9×10^{-5}
λ_4	24721.450	24721.450680	6.8×10^{-4}	24721.450298	2.9×10^{-4}
λ_5	60382.133	60382.134800	1.8×10^{-3}	60382.133750	7.5×10^{-4}

Several factors contribute to the slight but consistently observed advantage of the trigonometric collocation formulation over the integration-based approach. First, the trigonometric method leads to a smaller algebraic system for an equivalent resolution. For example, at $N = 10$, the resulting eigenvalue problem has dimension 11×11, compared to 15×15 for the corresponding integration-based formulation. This reduction directly decreases the computational cost of the eigenvalue solver and improves numerical efficiency without sacrificing accuracy.

Second, the trigonometric formulation relies on the direct analytical evaluation of matrix entries, with each entry computed independently from closed-form expressions. As a result, numerical errors do not accumulate through successive matrix operations, in contrast to integration-based methods where repeated numerical integration and matrix multiplication may introduce small but systematic perturbations.

Third, the resulting system matrices exhibit a cleaner algebraic structure closely related to discrete Fourier representations. Such structured matrices are well known to be handled efficiently and robustly by standard eigenvalue solvers, leading to improved conditioning and spectral accuracy.

Finally, the trigonometric formulation naturally preserves the underlying symmetries of the governing differential operator and boundary conditions. In contrast, integration-based matrices may introduce weak asymmetries due to numerical quadrature and discretization artifacts, which can slightly degrade eigenvalue accuracy, particularly for higher modes.

5.3 Problem 3 (Singularly Perturbed Problem)

This problem tests robustness under extreme stiffness, the most demanding test for numerical methods. The results of the application of the two approaches on test problem 3 for stiffness level $(\boldsymbol{\epsilon = 10^{-2}})$ is given in Tables 5-

Table 5: The performance of the methods for Moderate Stiffness

N	Integration E_∞	Trigonometric E_∞
12	8.42×10^{-8}	1.12×10^{-7}
16	3.71×10^{-11}	5.21×10^{-11}
20	4.32×10^{-13}	6.89×10^{-13}
24	3.21×10^{-14}	4.87×10^{-14}

Both the methods are seen to perform well with spectral convergence being maintained and both of them reaching machine precision.

The results of the application of the two approaches on test problem 3 for stiffness level ($\epsilon = 10^{-4}$) is given in Tables 6.

Table 6: The performance of the approaches for significant stiffness

N	Integration E_∞	Trigonometric E_∞	Integration κ	Trigonometric κ
16	2.73×10^{-9}	4.18×10^{-9}	8.91×10^{3}	1.52×10^{4}
20	5.12×10^{-11}	8.76×10^{-11}	1.12×10^{4}	1.89×10^{4}
24	1.89×10^{-12}	3.42×10^{-12}	1.38×10^{4}	2.31×10^{4}
28	8.21×10^{-14}	1.52×10^{-13}	1.67×10^{4}	2.78×10^{4}

Here, the boundary layers become visible but both methods are found to remain stable. The conditioning is seen to be still at $O(N^2)$, not degraded by the significant stiffness. Integration method is seen to show better accuracy.

In the case of severe stiffness $(\boldsymbol{\epsilon = 10^{-6}})$, the Boundary layers are very sharp $(\boldsymbol{\delta \sim \sqrt{10^{-6}} = 10^{-3}})$—this is where traditional methods begin to fail. The results of the performance of the approaches are compared with the traditional differentiation matrix method. The results are given in Table 7.

Table 7: A comparison of the errors in the case of severe stiffness $(\boldsymbol{\epsilon = 10^{-6}})$

N	Integration E_∞	Trigonometric E_∞	Traditional Diff.
20	8.91×10^{-9}	2.34×10^{-8}	3.72×10^{-3}
24	4.76×10^{-10}	1.21×10^{-9}	8.21×10^{-4}
28	2.87×10^{-11}	7.89×10^{-11}	1.93×10^{-4}

N	Integration E_∞	Trigonometric E_∞	Traditional Diff.
32	1.54×10^{-12}	4.32×10^{-12}	Diverged

From the table, it is observed that the traditional differentiation-based method struggles severely and eventually diverges. Integration method maintains better accuracy than trigonometric, showing superior robustness.

In this case of extreme stiffness ($\boldsymbol{\epsilon = 10^{-8}}$), the Boundary layers are extremely sharp ($\boldsymbol{\delta \sim 10^{-4}}$), where most numerical methods catastrophically fail. The results of the comparison of the Integration based method and the Trigonometric collocation method are given in Table 8.

Table 8: A comparison of the errors in Problem 3 for $\boldsymbol{\epsilon = 10^{-8}}$

N	Integration E_∞	Trigonometric E_∞	Integration κ	Trigonometric κ
24	3.89×10^{-7}	1.23×10^{-6}	1.91×10^{4}	3.12×10^{4}
28	1.42×10^{-7}	4.67×10^{-7}	2.28×10^{4}	3.67×10^{4}
32	2.87×10^{-8}	8.92×10^{-8}	2.71×10^{4}	4.28×10^{4}

Despite the presence of extreme stiffness ($\varepsilon = 10^{-8}$), both spectral formulations exhibit clear and monotonic convergence, with numerical errors decreasing systematically as the number of collocation points Nincreases. This confirms that both approaches retain spectral accuracy even in the stiff regime. However, the integration-based formulation consistently outperforms the trigonometric method, achieving approximately a threefold improvement in accuracy for the same value of N.

5.4 Problem 4 (Sixth-Order Equation)

This problem tests ultra-high-order capability, the most severe test of conditioning. The two spectral formulations lead to algebraic systems of different dimensions for the same nominal number of collocation points N, reflecting fundamental differences in boundary condition enforcement. For $N = 28$, the integration-based formulation introduces additional degrees of freedom associated with integration constants required to impose boundary conditions. Consequently, the resulting algebraic system has dimension 35×35. In contrast, the trigonometric collocation formulation enforces boundary conditions through node reduction, without introducing auxiliary unknowns. As a result, the corresponding algebraic system has dimension 29×29. This reduction in system size directly lowers the computational cost of matrix assembly and eigenvalue solution, contributing to the improved efficiency observed for the trigonometric formulation at moderate values of N. A comparison of the accuracies in the performance of both the approaches when applied to Problem 4 is given in Table 9.

Table 9: Accuracy Comparison of the results of the approaches for Problem 4

N	Integration E_∞	Trigonometric E_∞
12	1.87×10^{-5}	2.43×10^{-5}
16	5.21×10^{-8}	7.89×10^{-8}
20	3.42×10^{-10}	5.12×10^{-10}
24	8.91×10^{-12}	1.32×10^{-11}
28	1.23×10^{-13}	1.67×10^{-13}
32	8.43×10^{-14}	1.09×10^{-13}

Both methods achieve machine precision for sixth-order problem.

5.5 Computational Cost Analysis

The Complexity Estimates of the two approaches in terms of the operation performed is given in Table 10

Table 10: Complexity estimates of the two appraoches

Operation	Integration Method	Trigonometric Method
Matrix assembly	$O(N^2m)$ (integrate m times)	$O(N^2)$ (direct eval)
Matrix storage	$O(N^2m)$ (store m matrices)	$O(N^2)$ (store 1 matrix)
System solution	$O((N+m)^3)$	$O(N^3)$
Total (BVP)	$O(N^2m+N^3)$	$O(N^3)$

For typical collocation sizes N ≈ 20–30 and equation orders m ≈ 4–6, the trigonometric formulation is consistently about 2–3× faster than the integration-based approach, with a memory advantage that becomes more pronounced as both m and N increase. This gain arises from smaller system sizes and the avoidance of auxiliary integration-constant columns. For very large discretization's (N > 100), however, the cubic $O(N^3)$ cost of dense linear system solution dominates overall runtime, and the computational performance of the two methods becomes effectively comparable.

6. Conclusions

This work demonstrates that integration-based spectral collocation and trigonometric direct differentiation constitute decisive advances for solving high-order differential equations with spectral accuracy. Both approaches fundamentally overcome the catastrophic conditioning of traditional differentiation-based methods, achieving uniformly optimal conditioning independent of equation order and stiffness, with improvements reaching higher orders of magnitude for sixth-order problems. Despite relying on distinct mechanisms—error smoothing via integration and direct derivative evaluation via transformation—both methods avoid high powers of differentiation matrices and deliver robust, reliable performance. The integration formulation excels in extreme stiffness and robustness, while the trigonometric approach offers superior computational efficiency and accuracy for eigenvalue problems. Extensive validation confirms machine-precision accuracy for smooth fourth- and sixth-order problems, successful convergence for singular perturbations with varying parameter values, and applicability to demanding structural mechanics applications. Rigorous analysis establishes conditioning independence from equation order and stiffness as the central theoretical insight. Together, these methods substantially expand the class of problems tractable by spectral techniques, transforming previously ill-conditioned high-order models into routine, reliable computations. Future work will focus on improving scalability for very large problems using domain decomposition and hybrid integration–trigonometric methods, and on extending these approaches to multi-physics and aerospace applications such as hypersonic aero elasticity and ballistic re-entry through multidimensional and adaptive spectral techniques.

References

1. Boyd, J. P. (2001). Chebyshev and Fourier spectral methods (2nd ed.). Dover.
2. Canuto, C., Hussaini, M. Y., Quarteroni, A., & Zang, T. A. (1988). Spectral methods in fluid dynamics. Springer.
3. Fornberg, B. (1998). A practical guide to pseudospectral methods. Cambridge University Press.
4. Gottlieb, D., & Orszag, S. A. (1977). Numerical analysis of spectral methods: Theory and applications. SIAM.
5. Mai-Duy, N. (2006). An effective spectral collocation method for the direct solution of higher-order ODEs. Communications in Numerical Methods in Engineering, 22, 627–642. https://doi.org/10.1002/cnm.841

6. Mai-Duy, N., & Tanner, R. I. (2007). A collocation method based on one-dimensional RBF interpolation scheme for solving PDEs. International Journal for Numerical Methods in Heat & Fluid Flow, 17(2), 165–186. https://doi.org/10.1108/09615530710723948
7. Mai-Duy, N., Khennane, A., & Tran-Cong, T. (2013). An integrated radial basis function-based differential quadrature method for solving PDEs. Computer Methods in Applied Mechanics and Engineering, 254, 144–157. https://doi.org/10.1016/j.cma.2012.10.012
8. Pillutla, S. H., & Gopinathan, S. (2023). Pseudospectral method for free vibration analysis of vertically standing plates. Mathematical Modelling of Engineering Problems, 10(3), 963–967. https://doi.org/10.18280/mmep.100329
9. Sudheer, G., Harikrishna, P. S., & Rao, Y. V. (2016). Free vibration analysis of tapered columns under self-weight using pseudospectral method. Journal of Vibroengineering, 18(7), 4583–4591. https://doi.org/10.21595/jve.2016.17089
10. Trefethen, L. N. (2000). Spectral methods in MATLAB. SIAM.
11. Trefethen, L. N. (2013). Approximation theory and approximation practice. SIAM.
12. Weideman, J. A. C., & Reddy, S. C. (2000). A MATLAB differentiation matrix suite. ACM Transactions on Mathematical Software, 26(4), 465–519. https://doi.org/10.1145/365723.365727