Strong Fuzzy Detour μ – Centre in Fuzzy Graphs

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Abstract: The strong fuzzy detour μ-distance $\Delta_s(u, v)$ between two vertices of a connected fuzzy graph G is defined as the maximum of the μ-length of all strong paths connecting u and v. We introduce the strong fuzzy detour μ-eccentricity, μ-radius, μ-diameter, and μ-center notions. Based on these concepts, we establish some properties.

Keywords: connected, fuzzy, maximum

1.Introduction

Zadeh published a seminal article on fuzzy sets [10] in 1965, which led to fuzzy logic and, consequently, fuzzy set theory. A major goal of Zadeh's article was to develop a theory that could be applied to ambiguity and imprecision in human thinking, particularly in pattern recognition, information communication, and abstraction. According to this theory, the grade of membership of an element in a subset of a universal set should be a real number in the closed interval [0, 1].

In 1975, Rosenfeld studied fuzzy relations on fuzzy sets and created fuzzy graphs, which are analogous to graph theories. The concepts of connectedness, paths, bridges, clusters, trees, forests, and cut vertices were introduced and analyzed. In his work, P.Bhattacharya [1] examined automorphism of fuzzy graphs and introduced the concept of centre and eccentricity in fuzzy graphs. Strong arcs in fuzzy graphs and M - strong fuzzy graphs were the topics of discussion for K.R.Bhutani [3] and other researchers.

In fuzzy graphs, the notion of μ-distance was first presented by Rosenfled [7].

The fuzzy detour μ – distance and fuzzy detour μ –centre in fuzzy graph were discussed by A.Nagoorgani and J.Umamaheswari [6,7]. In this work, strong fuzzy μ – distance and strong fuzzy detour μ – centre have been introduced and analogous properties have been presented. A few basic concepts will be reviewed for this purpose.

2. Preliminaries

A **fuzzy graph** $G = (V, \sigma, \mu)$ is a nonempty set together with a pair of functions

$\sigma : V \to [0,1]$and $\mu : V \times V \to [0,1]$ such that for all x, y in V, $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$. We call σ the *fuzzy vertex set* of G and μ the *fuzzy edge set* of G, respectively. Note that μ is a fuzzy relation on σ. We will assume that, unless otherwise specified, the underlying set is V and that it is finite. For convenience, we use the notation $G = (\sigma, \mu)$ to represent the fuzzy graph $G = (V, \sigma, \mu)$.

A fuzzy graph $G = (\sigma, \mu)$ has a path P that is a sequence of unique vertices $u_0, u_1, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, \ldots, n$. The path's length is indicated here, n. The definition of P's strength is $\wedge_{i=1}^{n} \mu(u_{i-1}, u_i)$. In other words, the weight of the weakest edge determines a path's strength. $\mu^{\infty}(u,v)$ or $CONN_G(u,v)$ represent the strength of connectedness between two vertices, which is defined as the maximum of the strength of all paths between u and v. Any two vertices u, v can be joined by the strongest path, which has strength $\mu^{\infty}(u,v)$. A path connecting two vertices makes them connected.

Consider a fuzzy graph $G = (\sigma, \mu)$. In case $\mu(xy) > 0$ and $\mu(xy) \geq CONN_{G-xy}(x,y)$, an edge xy is considered strong in G. If for every $1 \leq i \leq n.$, the path : $u = u_0, u_1, \ldots, u_n = v$ from u to v is strong, then it is referred to as a strong path. Let x, y be two different vertices of a fuzzy graph $G = (\sigma, \mu)$, and let G' be the partial fuzzy subgraph of G that results from removing the edge (x,y). $G'(\sigma, \mu')$, in other words, where $\mu'(x,y) = 0$ and $\mu' = \mu$ for all other pairs. If $\mu'^{\infty}(u,v) < \mu^{\infty}(u,v)$ for some u,v, then (x,y) is a bridge in G. Put otherwise, if removing the edge (x,y) weakens the connection between a certain pair of vertices. Consequently, (x,y) is a bridge if and only if (x,y) is an edge of every strongest path connecting vertices u, v. For each u,v such that $u \neq w \neq v$, we say that w is a cutvertex in G if $\mu'^{\infty}(u,v) < \mu^{\infty}(u,v)$.

Stated differently, if the strength of connectivity between any other pair of vertices is diminished by eliminating vertex w. Hence, if and only if vertices u, separate from w exist such that w is on every strongest path from u to v, then w is a cutvertex. In this case, G' is either a block or non separable. If G doesn't have any fuzzy cutvertices, it is referred to as a fuzzy block.

The **$\boldsymbol{\mu}$ – distance**[4]$\delta(u,v)$is the smallest μ – length of any $u - v$ path, where the

μ – length of a path $P : u = u_0, u_1, \ldots, u_n = v$ is $L(P) = \sum_{i=1}^{n} \frac{1}{\mu(u_{i-1}, u_i)}$. The **detour**

$\boldsymbol{\mu}$ – distance[6] $\Delta(u,v)$is the longest μ – length of any $u - v$ path, where the μ – length of a path $P : u = u_0, u_1, \ldots, u_n = v$ is $L(P) = \sum_{i=1}^{n} \frac{1}{\mu(u_{i-1}, u_i)}$.

In our discussion, we prefer only the strong paths for which the fuzzy detour μ – distance is defined. That is, the **fuzzy detour $\boldsymbol{\mu}$ – distance for strong paths** is defined to be that distance given by $L(P) = \sum_{i=1}^{n} \frac{1}{\mu_s(u_{i-1}, u_i)}$where $P{:}u = u_0, u_1, \ldots, u_n = v$ and is denoted by $\Delta_s(u,v)$. This can also be called **strong fuzzy detour $\boldsymbol{\mu}$ – distance.**That is, the strong fuzzy detour

μ – distance $\Delta_s(u,v)$ between the vertices u and v is defined as the maximum of the μ - lengths of all the strong paths joining u and v.

The maximum of the strong fuzzy detour μ - distances from x to any vertex of G is the strong fuzzy detour μ - eccentricity, $e_{\Delta_s}(x)$, of a vertex x of a fuzzy graph G. The minimum of the strong fuzzy detour μ - eccentricities among the vertices of G is the strong fuzzy detour μ - radius of G, or $rad_{\Delta_s}(G)$. The maximum of the strong fuzzy detour μ - eccentricities among the vertices of G is the strong fuzzy detour μ - diameter of G or $diam_{\Delta_s}(G)$.

If $e_{\Delta_s}(v) = rad_{\Delta_s}(G)$, then a node v in a connected fuzzy graph G is a strong fuzzy detour μ – central node. The fuzzy subgraph created by the strong fuzzy detour μ – central nodes of G is referred to as the strong fuzzy detour μ – center of G and is symbolically represented as $C_{\Delta_s}(G)$.

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If $e_{\Delta_s}(v) = diam_{\Delta_s}(G)$, then node v in a connected fuzzy graph G is a strong fuzzy detour μ – peripheral node. The fuzzy subgraph created by the strong fuzzy detour μ – peripheral nodes of G is referred to as the strong fuzzy detour periphery of G, and it is symbolically represented by $Per_{\Delta_s}(G)$. A node v is said to be annular node if $rad_{\Delta_s}(G) < e_{\Delta_s}(v) < diam_{\Delta_s}(G)$.

A **complete fuzzy graph** is a fuzzy graph $G = (\sigma, \mu)$ such that $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in \sigma^*$. If $G = (\sigma, \mu)$ is a complete fuzzy graph, then $\mu^\infty = \mu$.

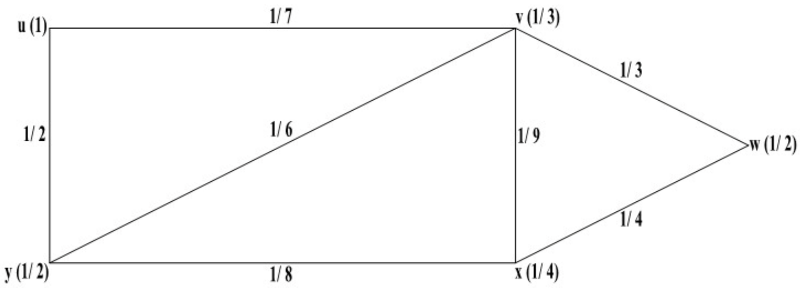

Fig. 2.1Fuzzy graph with strong edges

Example 2.1 Let $G = (\sigma, \mu)$ be a fuzzy graph with $\sigma^* = \{u, v, w, x, y\}. \sigma(u) = 1, \sigma(v) = 1/3$,

$\sigma(w) = \sigma(y) = 1/2$, $\sigma(x) = 1/4$,$\mu(uv) = 1/7$, $\mu(uy) = 1/2$,$\mu(vw) = 1/3$,

$\mu(vx) = 1/9$, $\mu(vy) = 1/6$, $\mu(wx) = 1/4$ and $\mu(xy) = 1/8$. Note that, the edges (u,y),(v,y),(v,w),(x,w) in G are strong and the strong fuzzy detour μ – distance between two vertices are as follows.

$\Delta_s(u, y) = 2, \Delta_s(u, v) = 8, \Delta_s(u, w) = 11$, $\Delta_s(v, x) = 7$,$\Delta_s(u, x)$ =15, $\Delta_s(v, y) = 6$, $\Delta_s(v, w) = 3$, $\Delta_s(x, y)$ =13, $\Delta_s(x, w) = 4$,$\Delta_s(y, w)$ =9, $e_{\Delta_s}(u) = e_{\Delta_s}(x) = 15$, $e_{\Delta_s}(v) = 8$, $e_{\Delta_s}(w) = 11$, $e_{\Delta_s}(y) = 13$.

Thus, $rad_{\Delta_s}(G) = 8$, $diam_{\Delta_s}(G) = 15$. Here, v is the central node, peripheral nodes are u and x and annular nodes are w and y.

3. Metrics in Strong Fuzzy Graphs

Theorem 3.1 *In a connected fuzzy graph G,* $\Delta_s(u, v)$ *is a metric.*

Proof : (i)$\Delta_s(u, v) = 0$ if and only if $u = v$ because $L(P) = 0$ if and only if the $u - v$ path P has length 0.

(ii)$\Delta_s(u, v) = \Delta_s(v, u)$ because the reversal of a path is a path and μ is symmetric.

It remains only to show that strong fuzzy detour μ – distance satisfies the triangle inequality, namely, $\Delta_s(u, v) \leq \Delta_s(u, w) + \Delta_s(w, v)$, where w is a vertex on the $u - v$ path.

Case(i) Let P_1 be the subpath of P from u to w and P_2 be the subpath of P from w to v. Then, $\Delta_s(u, w) = L(P_1)$ and $\Delta_s(w, v) = L(P_2)$. Now, $L(P) = L(P_1) + L(P_2)$. Therefore,

$\Delta_s(u, v) = \Delta_s(u, w) + \Delta_s(w, v)$.

Case (ii) Suppose the vertex w does not lie on the path P. Let $\Delta_s(w, x) = L(R)$. Then

$\Delta_s(u, w) \geq L(P_1) + L(R)$and $\Delta_s(w, v) \geq L(P_2) + L(R)$. Therefore, $\Delta_s(u, v) = L(P)$

$= L(P_1) + L(P_2) < L(P_1) + L(R) + L(P_2) + L(R) \leq \Delta_s(u, w) + \Delta_s(w, v)$. Thus by *cases (i)* and *(ii)* we get the required triangular inequality.

Theorem 3.2 Let δ_s and Δ_s represent the μ - distance and detour μ – distance respectively in a fuzzy graph G. Then, for any two vertices u and v in G, $0 \leq \delta_s(u, v) \leq \Delta_s(u, v) < \infty$.

Proof : Let $u, v \in G$. Since $\delta_s(u, v)$ is the shortest μ – distance and $\Delta_s(u, v)$ is the longest

μ – distance, it is obvious that $\delta_s(u, v) \leq \Delta_s(u, v)$ and the result follows immediately.

Corollary 3.3 If u and v are any two vertices in a connected fuzzy graph G, then $\Delta_s(u, v) = 0$ if and only if $\delta_s(u, v) = 0$.

Theorem 3.4 If G is a connected fuzzy graph, then for any vertex $v \in G$,

$diam_{\Delta_s}(G) - e_{\Delta_s}(v) \geq t$ where t is any nonnegative real number.

Proof : For any $v \in G$, $e_{\Delta_s}(v) = max\{\Delta_s(v, x)/ x \in G\}$ and $diam_{\Delta_s}(G) = max\{e_{\Delta_s}(v)/ v \in G\}$.

If $diam_{\Delta_s}(G) = e_{\Delta_s}(v)$, then $diam_{\Delta_s}(G) - e_{\Delta_s}(v) = 0$; on the other hand, if $diam_{\Delta_s}(G) > e_{\Delta_s}(v)$, then $diam_{\Delta_s}(G) - e_{\Delta_s}(v) > 0$. If this positive value is some real quantity, say t, then the result follows immediately.

The following result gives an important relationship between fuzzy detour μ – radius and

fuzzy detour μ –diameter for strong paths of a fuzzy graph which is analogous to the result for all paths in a fuzzy graph.

Theorem 3.5 For any connected fuzzy graph G, $rad_{\Delta_s}(G) \leq diam_{\Delta_s}(G) \leq 2\, rad_{\Delta_s}(G)$.

*Proof :*By the definition of strong fuzzy detour μ – radius and strong fuzzy detour μ –diameter, we can conclude that $rad_{\Delta_s}(G) \leq diam_{\Delta_s}(G)$. To prove the right hand side inequality, we choose a strong path P from u to v. Let $y \in P$ be an element of the strong fuzzy detour

μ –centre of G. Then, $e_{\Delta_s}(y) = rad_{\Delta_s}(G)$. If x and z be two strong fuzzy detour peripheral nodes of G, then $e_{\Delta_s}(x) = e_{\Delta_s}(z) = diam_{\Delta_s}(G)$. By triangle inequality, we have

$\Delta_s(x, z) \leq \Delta_s(x, y) + \Delta_s(y, z)$; that is, $diam_{\Delta_s}(G) \leq rad_{\Delta_s}(G) + rad_{\Delta_s}(G) = 2\, rad_{\Delta_s}(G)$.

Therefore, the result follows.

Next result shows how fuzzy detour μ – eccentricities of distinct vertices for strong fuzzy graphs can be related uniquely.

Theorem 3.6 If u and v are any two distinct vertices of a connected fuzzy graph G, then

$|e_{\Delta_s}(u) - e_{\Delta_s}(v)| \leq \Delta_s(u, v)$.

Proof: $\Delta_s(u, v)$ denotes the maximum of μ – distances between u and v such that each edge is strong in the path joining u and v. Then, by definition, $e_{\Delta_s}(u) = max\{\Delta_s(u, x)/ x \in G\}$. For some $y \in G$, let $e_{\Delta_s}(u) = \Delta_s(u, y)$. Then, $\Delta_s(v, y) \leq e_{\Delta_s}(v)$. By triangle inequality, we can write

$e_{\Delta_s}(u) = \Delta_s(u, y) \leq \Delta_s(u, v) + \Delta_s(v, y) \leq \Delta_s(u, v) + e_{\Delta_s}(v)$. That is, $e_{\Delta_s}(u) - e_{\Delta_s}(v) \leq \Delta_s(u, v)$.

Interchanging the roles of u and v, the above inequality can be written as

$e_{\Delta_s}(v) - e_{\Delta_s}(u) \leq \Delta_s(u, v)$. That is, $e_{\Delta_s}(u) - e_{\Delta_s}(v) \geq -\Delta_s(u, v)$. Combining these two inequalities, we obtain $-\Delta_s(u, v) \leq e_{\Delta_s}(u) - e_{\Delta_s}(v) \leq \Delta_s(u, v)$ which is the required result.

4. Strong Fuzzy detour μ – centre

A node v in a connected fuzzy graph G is a **strong fuzzy detour μ – central node** if $e_{\Delta_s}(v) = rad_{\Delta_s}(G)$ and the fuzzy subgraph induced by the strong fuzzy detour μ – central nodes of G is called the **strong fuzzy detour μ – centre** of G and is denoted symbolically by $C_{\Delta_s}(G)$.

A **complete fuzzy graph** is a fuzzy graph $G = (\sigma, \mu)$ such that $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in \sigma^*$. If $G = (\sigma, \mu)$ is a complete fuzzy graph, then $\mu^\infty = \mu$.

Theorem 4.1 The strong fuzzy detour μ – centre $C_{\Delta_s}(G)$ of every strong connected fuzzy graph G lies in a single block of G.

Proof: We prove this theorem by contradiction. Assume that G is a strong connected fuzzy graph whose strong fuzzy detour μ – centre does not lie in a single block of G. Then there exists a cut vertex w of G^* such that $G^* - w$ contains the blocks B_1 and B_2 which contain the elements u and v of $C_{\Delta_s}(G)$. Let $u \in B_1$ such that $\Delta_s(u, w) = e_{\Delta_s}(w)$. Let $v \in B_2$. Let $B_1$and B_2 contain respectively the $u - w$ strong fuzzy detour and the $w - v$ strong fuzzy detour. Then, $\Delta_s(u, v) > \Delta_s(u, w)$, because w, being a cutvertex, lies on every strong fuzzy detour. That is, $\Delta_s(u, v) > e_{\Delta_s}(w)$. Since $e_{\Delta_s}(v) \geq \Delta_s(u, v)$, we conclude that $e_{\Delta_s}(v) > e_{\Delta_s}(w)$. This contradicts the fact that v is a strong fuzzy detour central node of G.

If every vertex of a fuzzy graph is a strong fuzzy detour central node, then the fuzzy graph is called self-centered. The following example explains this concept in detail.

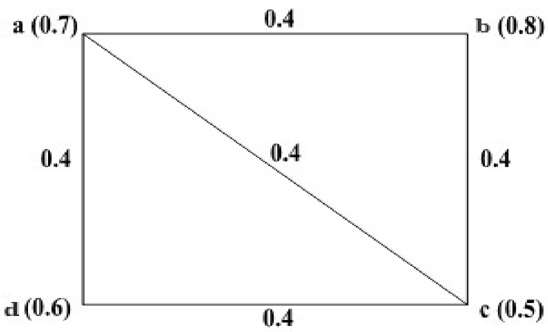

Fig. 4.1 Strong Fuzzy detour self-centered fuzzy graph

Example4.2 Let $V = \{a, b, c, d\}$. Define the fuzzy subsets σ of $V$and μ of $E = \{ab, ac, bc, cd, da\}$ as follows: $\sigma(a) = 0.7$, $\sigma(b) = 0.8$, $\sigma(c) = 0.5$, $\sigma(d) = 0.6$ and $\mu(ab) = \mu(ac) = \mu(bc) = \mu(cd) = \mu(da) = 0.4$. $\Delta_s(a, b) = \Delta_s(b, c) = \Delta_s(c, d) = \Delta_s(d, a)$

$= 7.5$ and $\Delta_s(a, c) = 5$. $e_{\Delta_s}(a) = e_{\Delta_s}(b) = e_{\Delta_s}(c) = e_{\Delta_s}(d) = 7.5$. $rad_{\Delta_s}(G) = 7.5$ and $diam_{\Delta_s}(G) = 7.5$. therefore, $C_{\Delta_s}(G) = \{a, b, c, d\}$; that is, all edges of the fuzzy graph are strong fuzzy detour central nodes. Thus, G is strong fuzzy detour self-centered graph.

5 Conclusions:

The idea of fuzzy detour μ – distance is introduced in fuzzy graphs having strong paths. The properties we have developed are analogous. This new approach may help to improve the facilities to areas which have importance of public service. To conclude, we note that it is possible to make use of the wealth of fuzzy graph theory which includes significant applications to data structures and algorithms analysis.

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